THE AMERICAN
MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION
OF AMERICA

VOLUME XXIV MARCH, 1917 NUMBER 3

DIRECTED ANGLES IN ELEMENTARY GEOMETRY.

BY R. A. JOHNSON, Western Reserve University.

In many of the theorems of the modern elementary geometry, the treatment of angles seems somewhat unsatisfactory. In the statement of theorems, one is often confronted with the dilemma of a choice between an inaccurate statement, and one so verbose and involved as to be unwieldy. Again, many proofs, as given in the texts,[1] are insufficient because they apply only to particular positions of the figure. A very common example is the following. If A, B, C, D are four points on a circle, the angles ABC, ADC are equal or supplementary, according as B and D are on the same side of AC, or on opposite sides. This theorem is repeatedly used in proofs; but in a given case, when we know only that the points are concyclic, and have no data as to their order on the circle, how are we to decide which of the two possibilities is the correct one? Apparently the usual custom is to draw a single figure, and decide by inspection of the figure, trusting that the proof so obtained can be modified to fit all possible figures. Not only is such a method entirely unscientific, but in cases where the figure is at all complicated, the determination of the number of possibilities and the corresponding modifications of the proof are practically impossible.

As a simple illustration, let us consider Simson's theorem, so-called. "If from any point P on the circumcircle of the triangle ABC, PX, PY, PZ be drawn perpendicular to the sides, the points X, Y, Z will be collinear." (This statement, and the following proof, are taken from Lachlan, *l. c.*, § 120.)

"Join ZX, YX. Then since the points P, X, Z, B are concyclic, the angle PXZ is the supplement of the angle ABP. And since P, Y, C, X are concyclic, the angle YXP is the supplement of the angle YCP, and is equal to the angle ABP, because P, C, A, B are concyclic."

[1] Such books as Lachlan, *Modern Pure Geometry;* Casey, *A Sequel to Euclid;* McClelland, *Geometry of the Circle;* etc., are here referred to.

"Hence the angles PXZ, YXP are supplementary, and therefore ZX, XY are in the same straight line."

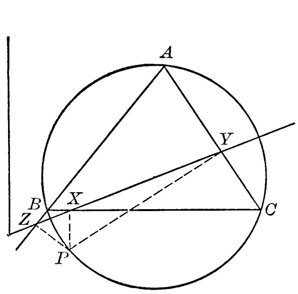

FIG. 1.

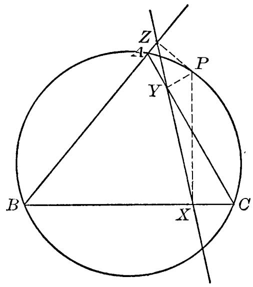

FIG. 2.

Now in Fig. 1, which resembles the one in the text, each of these assertions is true; but in Fig. 2, which illustrates equally well the theorem as stated, most of them are untrue; angles stated to be supplementary are equal in three different cases. Hence the proof fails to be adequate for all cases of the theorem. In fact, it will doubtless require some thought on the part of the reader to decide how many different cases Lachlan ought to consider; and this is an unusually simple example.

The scheme developed below has been devised to meet these difficulties; to make exact statements and general and rigorous proofs possible. So far as the writer is aware, such a method has not been used in elementary work, though the underlying ideas are obviously familiar ones in analytic geometry.

Definitions. Let us first agree to distinguish between a positive and a negative direction of rotation. Following the usual custom, we shall regard an angle as positive when it is generated by rotation in anti-clockwise direction, and negative in the other direction.

The symbol $\angle BAC$ shall mean the angle between the half-lines AB, AC, a signless quantity, just as in elementary geometry, so that

$$\angle BAC = \angle CAB.$$

If l, l' are any two lines, the symbol $\measuredangle\, l, l'$ shall mean *that angle through which l must be rotated in a positive direction in order that it may come to coincide with l'*. Similarly the symbol $\measuredangle BAC$ means *the positive angle through which the line AB, taken as a whole, must be rotated, to coincide with the line AC taken as a whole;* and this without regard to the position of B and C on these lines.

For example, if A, B, C are any points on a line, and D a point not on that line, we have, whatever the order of A, B, C,

$$\measuredangle ABD = \measuredangle CBD.$$

The quantity thus defined and denoted by $\measuredangle ABC$ is called *the directed angle from AB to BC.*

Further be it noted as an immediate consequence of the definition that two

directed angles are regarded as equivalent if they differ only by multiples of a straight angle.

The *addition* of directed angles is defined by the following laws, seen to be consistent with the definition. $\measuredangle\, l_1, l_2 + \measuredangle\, l_2, l_3 = \measuredangle\, l_1, l_3$; $\measuredangle\, l_1, l_2 + \measuredangle\, l_3, l_4 = \measuredangle\, l_1, l_5$, where l_5 is a line so located that $\measuredangle\, l_2, l_5 = \measuredangle\, l_3, l_4$.

From these definitions we have the following relations as bases of operations with directed angles

THEOREM I. $\measuredangle\, l, l' + \measuredangle\, l', l = 180°$.

THEOREM II. *If l_1 is parallel to l_1', and l_2 to l_2', then $\measuredangle\, l_1, l_2 = \measuredangle\, l_1', l_2'$.* Again, *if l_1 is* perpendicular *to l_1', and l_2 to l_2', then $\measuredangle\, l_1, l_2 = \measuredangle\, l_1', l_2'$.*

THEOREM III. *For any four lines $\measuredangle\, l_1, l_2 + \measuredangle\, l_3, l_4 = \measuredangle\, l_1, l_4 + \measuredangle\, l_3, l_2$.* For, $\measuredangle\, l_1, l_2 = \measuredangle\, l_1, l_4 + \measuredangle\, l_4, l_2$, and $\measuredangle\, l_3, l_4 = \measuredangle\, l_3, l_2 + \measuredangle\, l_2, l_4$.

THEOREM IV. *A necessary and sufficient condition that three points A, B, C lie on a line is that for any other point D we have*

$$\measuredangle\, ABD = \measuredangle\, CBD.$$

For, if AB and CB are equally inclined to BD, they coincide, and conversely.

THEOREM V. *The necessary and sufficient condition that four points A, B, C, D lie on a circle is that $\measuredangle\, ABD = \measuredangle\, ACD$*

For this equation means that (a) if B and C are on the same side of AD, then $\angle\, ABD$ and $\angle\, ACD$ are equal; and (b) if B and C are on opposite sides of AD, then $\angle\, ABD$ is equal to the supplement of $\angle\, ACD$. Hence the present theorem follows from the theorem quoted in the first paragraph above.

It would be hard to overestimate the importance of this last theorem. Let us illustrate by proving Simson's theorem, using the same notation as previously, but any figure which may be drawn.

Proof. Since PXB, PZB are right angles, P, B, X, Z lie on a circle (in what order we do not know nor care).

Hence, $\measuredangle\, PXZ = \measuredangle\, PBZ$.

Similarly, P, X, Y, C are concyclic, and $\measuredangle\, PXY = \measuredangle\, PCY$. But $\measuredangle\, PBX$ is identically the same as $\measuredangle\, PBA$, and $\measuredangle\, PCY$ the same as $\measuredangle\, PCA$. Hence $\measuredangle\, PXZ = \measuredangle\, PBA$, and $\measuredangle\, PZY = \measuredangle\, PCA$. But since A, B, C are concyclic, $\measuredangle\, PBA = \measuredangle\, PCA$, and $\measuredangle\, PXZ = \measuredangle\, PXY$, which, by theorem IV, shows that X, Y, Z are collinear.

It is obvious that this is an entirely general proof. It is a little more verbose than need be, in order to bring out the method clearly. We now apply similar methods to a few more well-known theorems.

THEOREM. *If a point is marked on each side of a triangle (or its extension), and the circles drawn, each of which passes through a vertex of the triangle and the points marked on the adjacent sides, these circles pass through a point, and the lines from this point to the marked points make equal angles with the sides.*

Let the triangle be $A_1A_2A_3$ (Fig. 3), let P_1, P_2, P_3 be marked on A_2A_3, A_3A_1, A_1A_2 respectively. Let circles $A_1P_2P_3$, $A_2P_3P_1$ be drawn, and meet at P.

Then

$$\measuredangle\, PP_2, PP_3 = \measuredangle\, P_2A_1P_3 = \measuredangle\, A_3A_1A_2,$$

$$\measuredangle\, PP_3, PP_1 = \measuredangle\, P_3A_2P_1 = \measuredangle\, A_1A_2A_3.$$

Adding,

$$\measuredangle\, PP_2, PP_1 = \measuredangle\, A_3A_1, A_1A_2 + \measuredangle\, A_1A_2, A_2A_3$$
$$= \measuredangle\, A_3A_1, A_2A_3 = \measuredangle A_1A_3A_2 = \measuredangle\, P_2A_3P_1.$$

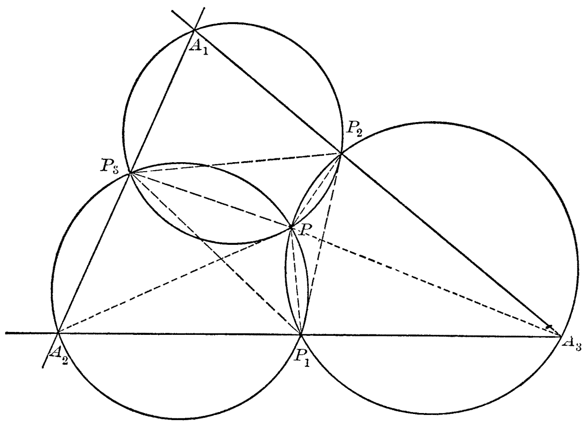

FIG. 3.

Whence, by theorem V, P, P_1, P_2, A_3 are concyclic, and the theorem is proved. Incidentally we see that $\measuredangle\, PP_1, A_2A_3 = \measuredangle\, PP_2, A_3A_1 = \measuredangle\, PP_3, A_1A_2$.

THEOREM. *In the same figure*, $\measuredangle\, A_2PA_3 = \measuredangle\, P_2P_1P_3 + \measuredangle\, A_2A_1A_3$.

The proof consists of splitting $\measuredangle\, A_2PA_3$ into two parts,

$$\measuredangle\, A_2PA_3 = \measuredangle\, A_2PP_1 + \measuredangle\, P_1PA_3.$$

Now

$$\measuredangle\, A_2PP_1 = \measuredangle\, A_2P_3P_1 = \measuredangle\, A_2A_1P_1 + \measuredangle\, A_1P_1P_3,$$

and

$$\measuredangle\, P_1PA_3 = \measuredangle\, P_1P_2A_3 = \measuredangle\, P_2P_1A_1 + \measuredangle\, P_1A_1A_3,$$

and we get the desired result by adding. Of course there are similar expressions for $\measuredangle\, A_3PA_1$ and $\measuredangle\, A_1PA_2$. To see the difficulties encountered by attacking this figure without due care, the reader should note McClelland, pages 40–41.

The above is called the theorem of Miquel.[1] The point is called the Miquel point for the set of points P_1, P_2, P_3.

COROLLARIES. (1) If P is a fixed point, it is the Miquel point of infinitely many triangles inscribed in $A_1A_2A_3$. These triangles are all directly similar, with P as center of similitude.

(2) If P lies on the circumcircle of $A_1A_2A_3$, then P_1, P_2, P_3 are collinear, and conversely (Simson's theorem).

[1] Cf. J. L. Coolidge, *Geometry of the Circle*, 1916, p. 85.

For $\measuredangle P_2P_1P_3 = 0$ if and only if $\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3$.

(3) Among the triangles having a given point P for Miquel point is the pedal triangle of P, *i. e.*, the triangle whose vertices are the feet of the perpendiculars from P to the sides of $A_1A_2A_3$. The angles of the pedal triangle of any point are therefore given by the formulas $\measuredangle P_3P_1P_2 = \measuredangle A_2A_1A_3 + \measuredangle A_3PA_2$, etc.

(4) Conversely, if three circles are concurrent at a point, it is possible in an infinite number of ways to draw a triangle having one vertex on each circle and one side passing through each of the intersections of the circles two by two. All such triangles are similar.

Other corollaries suggest themselves readily.

We close with another fundamental theorem, of much less importance, and an application of it to a rather difficult theorem of Steiner.

THEOREM VI. *If O is the center of the circle through A, B, C, then*

$$\measuredangle OAB = \measuredangle ACB + 90°.$$

For, let AO meet the circle again at D. By the rule for adding angles,

$$\measuredangle OAB = \measuredangle ADB + \measuredangle DBA = \measuredangle ACB + 90°.$$

Now corollary 2 above may be re-stated in the following familiar form:

THEOREM. *The circumcircles of the four triangles of a complete quadrilateral meet in a point.*

For if $P_1P_2P_3$ is a transversal of triangle $A_1A_2A_3$, we have seen that the four circles $A_1A_2A_3$, $A_1P_2P_3$, $A_2P_3P_1$, $A_3P_1P_2$ are concurrent. And obviously any complete quadrilateral may be regarded as a triangle and a transversal.

THEOREM (Steiner). *The centers of the four circumcircles lie on a circle which also passes through this point.*

Let P be the intersection of the four circles named above, and let their centers, in the order named, be O, C_1, C_2, C_3. Then $C_1O \perp A_1P$, $C_3O \perp A_3P$, and hence

$$\measuredangle C_1OC_3 = \measuredangle A_1PA_3 = \measuredangle A_1A_2A_3.$$

Similarly for C_2, and we see that the four centers are concyclic. To show that this circle passes through P is not so simple. The triangles C_1PC_3 and $AC_1P_2C_3$ lie symmetrically with regard to C_1C_3; hence

$$\measuredangle C_1PC_3 = \measuredangle C_3P_2C_1 = \measuredangle C_3P_2P_1 + \measuredangle P_3P_2C_1. \quad \text{(addition formula)}$$

But

$$\measuredangle C_3P_2P_1 = \measuredangle P_2A_3P_1 + 90°$$

and

$$\measuredangle P_3P_2C_1 = \measuredangle PA_1P_2 + 90°. \quad \text{(theorem VI)}$$

Hence

$$\measuredangle C_1PC_3 = \measuredangle P_2A_3P_1 + \measuredangle PA_1P_2 = \measuredangle P_3A_1,\, A_3P_1 = \measuredangle A_1A_2A_3 = \measuredangle C_1OC_3$$

and the proof is completed.